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THE GENERATION OF SURFACE WAVES BY A BUOYANT VORTEX RING†

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The perturbation of the free surface of an ideal, infinitely deep liquid located in a uniform gravitational field caused by the vertical motion of a thin vortex ring is found in the small-wave approximation.

LET A thin vortex ring of radius R and intensity Γ move normal to the free surface of an ideal ponderable fluid which occupies the half-space $z<0$. At all times, the centre of the vortex ring is located on the z axis of a cylindrical system of coordinates $\{r, \alpha, z\}$. If the intensity Γ is not too great and the vortex ring is located at a sufficient depth, then small-amplitude waves will develop on the free surface of the fluid. At the same time, the boundary conditions [1]

$$\Phi_t+gS=0,\qquad \Phi_z=S_t,\qquad \Phi_{tt}+g\Phi_z=0 \tag{1}$$

are satisfied in the plane $z=0$, where $S=S(r,\alpha,t)$ is the deviation of the free surface from its unperturbed position $z=0$.

We shall seek the velocity potential in the form $\Phi=\Phi_1+\varphi$, where Φ_1 is the velocity potential which is induced in an unbounded medium by a vortex ring of the same intensity as that being considered, which is symmetrically located with respect to the $z=0$ plane and the function φ is harmonic everywhere in the flow domain and is independent of the coordinate α by virtue of the symmetry of the problem

$$\Delta\varphi=\varphi_{rr}+r^{-1}\varphi_r+\varphi_{zz}=0 \tag{2}$$

It follows [2] from the explicit expression for the velocity potential which is induced by the thin vortex ring that $\Phi_1=0$ when $z=0$. If, at the initial instant of time, $t=0$, $S=0$ and $S_t=0$, then we find from the boundary conditions (1) that

$$\varphi_{tt}+g\varphi_z|_{z=0}=-gu(r;\ R,\ H),\qquad \varphi_z|_{z=0,\ t=0}=-u(r;\ R_0,H_0),\qquad \varphi_t|_{z=0,\ t=0}=0 \tag{3}$$

where

$$u=\left.\frac{\partial\Phi_1}{\partial z}\right|_{z=0}=\frac{\Gamma R}{2\pi}\int_0^{2\pi}\frac{(R-r\cos\theta)\,d\theta}{(H^2+R^2+r^2-2rR\cos\theta)^{3/2}}$$

is the vertical component of the velocity which is induced in the $z=0$ plane by vortex rings of intensity Γ with their centres at the points $z=\pm H$, and R_0 and H_0 are the radius of a vortex ring and the distance from it up to the surface of the fluid when $t=0$.

Let us now apply a zeroth order Fourier–Bessel transformation to the function φ

$$F(\rho,z)=\int_0^\infty\varphi(r,z)J_0(r\rho)r\,dr$$

Equation (2) is then transformed into the equation $F_{zz}-\rho^2F=0$. Substitution of its solution, $F=B(\rho,t)\exp(\rho z)$, which decays when $z\to-\infty$, into the Fourier–Bessel transformed conditions (3) leads to the problem

$$B_{tt}+g\rho B=-gU,\qquad B|_{t=0}=-U_0/\rho,\qquad B_t|_{t=0}=0$$

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where U is the Fourier–Bessel transform of the velocity u and $U_0 = U|_{t=0}$. The solution of this problem can be written in the form

$$B = \frac{1}{\rho}\left\{-U + \int_0^t U_\xi(\rho; R(\xi), H(\xi))\cos[\sqrt{g\rho}(t-\xi)]\,d\xi\right\} \tag{4}$$

Using the inverse Fourier–Bessel transformation, we find

$$\varphi=\varphi_1+f$$

$$\varphi_1=-\int_0^\infty U(\rho; R, H)\exp(\rho z)J_0(r\rho)\,d\rho \tag{5}$$

$$f=\int_0^\infty \exp(\rho z)J_0(r\rho)\int_0^t U_\xi(\rho; R(\xi), H(\xi))\cos[\sqrt{g\rho}(t-\xi)]\,d\xi\,d\rho$$

Everywhere in the flow domain, $\Delta\varphi_1 = 0$ and $\partial\varphi_1/\partial z|_{z=0} = -u$. On the other hand, the velocity potential Φ_2, which is induced in the lower half-space by a vortex ring of intensity —2Γ with its centre at a point $z = H$, is a harmonic function when $z<0$, which satisfies the boundary condition $\partial\Phi_2/\partial z|_{z=0} = -u$. It follows from this that the velocity potential of the entire flow $\Phi = \Phi_0+f$, where Φ_0 is the velocity potential induced by the vortex ring of intentity—Γ being considered, which is located symmetrically about the $z = 0$ plane, and the second term f is directly associated with the development of surface waves.

Hence, the velocity of the vortex ring which moves normal to the free surface consists of a self-induced velocity [1] $V = [\Gamma/(4\pi R)][\ln(8R/a-0.25)]$ (a is the radius of the vortex tube of the ring), the velocity induced by a "mirror" ring with an intensity of the opposite sign, and the velocity in the bulk of the fluid which is caused by the motion of its free surface.

The expression for the shape of the free surface

$$S=-\frac{1}{g}\frac{\partial\varphi}{\partial t}\bigg|_{z=0}=-\frac{1}{g}\int_0^\infty B_t(\rho,t)J_0(r\rho)\,\rho\,d\rho$$

follows from (1). When account is taken of (4), this can be written in the form

$$S=\frac{1}{\sqrt{g}}\int_0^\infty \sqrt{\rho}\,J_0(r\rho)\int_0^t U_\xi(\rho; R(\xi), H(\xi))\sin[\sqrt{g\rho}(t-\xi)]\,d\xi\,d\rho \tag{6}$$

If the vortex ring is located at a sufficiently great depth ($H\gg R$), the effect of the "mirror" vortex and the surface waves on its motion can be neglected and it can be assumed that it occurs at a constant velocity V when $R = R_0 = \text{const}$ according to the law $H(t) = H_0 - Vt$. Under these conditions, $u\approx\Gamma R_0^2(H^2+r^2)^{-3/2}$ which corresponds to the Fourier–Bessel transform [3] $U = \Gamma R_0^2\exp(-\rho H)/H$, while expression (6) becomes

$$S=\frac{\Gamma V R_0^2}{\sqrt{g}}\int_0^\infty \sqrt{\rho}\,J_0(r\rho)\int_0^t \frac{1+\rho H(\xi)}{H^2(\xi)}\exp[-\rho H(\xi)]\sin[\sqrt{g\rho}(t-\xi)]\,d\xi\,d\rho \tag{7}$$

By applying the mean value theorem to the inner integral (7) and making use of the asymptotic formulas in [4], we find that

$$S=\Gamma V R_0^2 t t_* \frac{6H(t_*)+g t_*^2}{6H^2(t_*)}\frac{1}{r^3}+o\left(\frac{1}{r^3}\right)\quad r\to\infty$$

where $t_*\in[0, t]$ is a certain intermediate instant of time.

Profiles of the free surface at $t = 1, 2$, and 3 seconds (curves 1–3 respectively) found from formula (7) using a digital computer are shown in Fig. 1 for the case of a vortex ring with the parameters: $R = 5$ cm, $a = 0.1$ mm, $\Gamma = 2.5\pi\times10^{-3}$ m^2/s ($V\approx0.1$ m/s) which is located at the initial instant of time at a depth $H_0 = 1$ m. The development of surface waves occurs against a background of a rapidly increasing elevation of the free surface in the very large neighbourhood of the point $r = 0$ [for example, $S(0)\approx32\times10^{-7}$ m when $t = 5$ s]. This may be explained by the occurrence of a vertical flow which is created by the vortex ring during its upward motion.

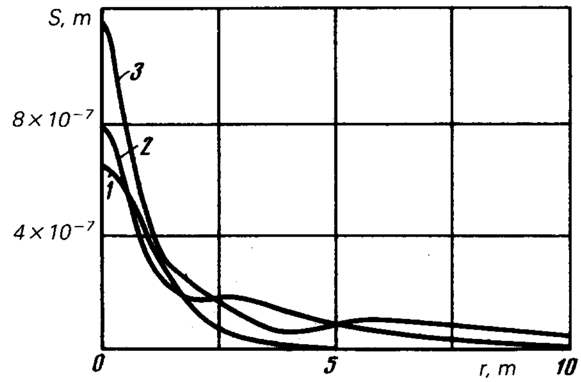

FIG. 1.

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